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Smarr's Formula for Black Holes in String Theory

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Abstract

We investigate thermodynamical properties of four and five dimensional black hole solutions of toroidally compactified string theory. We derive an analog of Smarr's formula, and verify it directly using the metric.

1 Introduction

Smarr's formula [1] has an important place in studies of black hole thermodynamics. It gives us the variation of mass in terms of the variations in entropy, angular momentum and electromagnetic charges, and provides us with an analogous thermodynamic relationship for the black hole system.

If we have a black hole solution, we can easily calculate the surface area of the outer horizon using the metric components. Entropy is given as $S = \frac{1}{4G_N}A$. Then, if the algebraic equations are tractable, we can isolate M, take derivatives, and obtain Smarr's formula.

This procedure won't work if the solution is given in terms of parameters that can't be solved explicitly in terms of mass and charges. Then we have to use a roundabout way, using infinitesimal variations to make a change of variables, which, in general involve inverting a big matrix, and if all entries are nonzero, the results may be too complicated.

We expect the variation of mass with respect to area to be the temperature of the black hole, and the variation of mass with respect to angular momentum to be the angular velocity. These quantities can be computed from the metric. This will be an independent way to calculate the coefficients in the Smarr's formula, and the results can be checked.

In this paper, we will follow the summarized procedure for two different types of black holes corresponding to four and five dimensional solutions of toroidally compactified string theory. We first write the area in terms of solution parameters, take the infinitesimal variation of the area, and replace the solution parameters by the physical ones using the jacobian matrix. Then we calculate Ω and κ using the metric, and compare the results with the Smarr's formula.

In Section II, a four dimensional rotating black hole parametrized by ADM mass, four charges, and angular momentum [2], and in Section III a five dimensional black hole with two angular momenta and three charges will be analyzed. [3]

Some of the related results for black holes in string theory have been given in [4],[5],[6]. A novel approach can be found in [7].

2 Four Dimensions

2.1 The Metric and Physical Parameters

The metric for four-dimensional rotating charged black hole solutions of toroidally compactified superstring theory, parametrized by the ADM mass, four charges and angular momentum, is given by [2]

$$
\begin{aligned}
ds_E^2 &= \Delta^{\frac{1}{2}}[-\frac{r^2-2mr+l^2\cos^2\theta}{\Delta}dt^2+\frac{dr^2}{r^2-2mr+l^2}+d\theta^2+\frac{\sin^2\theta}{\Delta}\{(r+2m\sinh^2\delta_1) \\
&\times (r+2m\sinh^2\delta_2)(r+2m\sinh^2\delta_3)(r+2m\sinh^2\delta_4)+l^2(1+\cos^2\theta)r^2+W \\
&+ 2ml^2r\sin^2\theta\}d\phi^2-\frac{4ml}{\Delta}\{(\cosh\delta_1\cosh\delta_2\cosh\delta_3\cosh\delta_4 \\
&- \sinh\delta_1\sinh\delta_2\sinh\delta_3\sinh\delta_4)r+2m\sinh\delta_1\sinh\delta_2\sinh\delta_3\sinh\delta_4\}\sin^2\theta dt d\phi], \qquad (1)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta &\equiv (r+2m\sinh^2\delta_1)(r+2m\sinh^2\delta_2)(r+2m\sinh^2\delta_3)(r+2m\sinh^2\delta_4) \\
&+ (2l^2r^2+W)\cos^2\theta, \\
W &\equiv 2ml^2(\sinh^2\delta_1+\sinh^2\delta_2+\sinh^2\delta_3+\sinh^2\delta_4)r \\
&+ 4m^2l^2(2\cosh\delta_1\cosh\delta_2\cosh\delta_3\cosh\delta_4\sinh\delta_1\sinh\delta_2\sinh\delta_3\sinh\delta_4 \\
&- 2\sinh^2\delta_1\sinh^2\delta_2\sinh^2\delta_3\sinh^2\delta_4-\sinh^2\delta_1\sinh^2\delta_2\sinh^2\delta_4 \\
&- \sinh^2\delta_1\sinh^2\delta_2\sinh^2\delta_3-\sinh^2\delta_2\sinh^2\delta_3\sinh^2\delta_4-\sinh^2\delta_1\sinh^2\delta_3\sinh^2\delta_4) \\
&+ l^4\cos^2\theta. \qquad (2)
\end{aligned}
$$

The outer and inner event horizons are at

$$
r_\pm = m \pm \sqrt{m^2-l^2}, \qquad (3)
$$

Here, m, the non-extremality parameter, is related to the mass of Kerr solution, l is related to the angular momentum of the Kerr solution and $\delta_{1,2,3,4}$ are boost parameters. Our aim is to write the variation of S in terms of the physical parameters ADM mass M, two electric charges Q_1, Q_2, two magnetic charges P_1, P_2, and the angular momentum J. The physical parameters can be expressed in terms of m, l and the boosts as follows:

$$
\begin{aligned}
M &= 4m(\cosh^2\delta_1+\cosh^2\delta_2+\cosh^2\delta_3+\cosh^2\delta_4)-8m, \\
Q_1 &= 4m\cosh\delta_1\sinh\delta_1, \\
Q_2 &= 4m\cosh\delta_2\sinh\delta_2, \qquad (4) \\
P_1 &= 4m\cosh\delta_3\sinh\delta_3, \\
P_2 &= 4m\cosh\delta_4\sinh\delta_4, \\
J &= 8\,l\,m\,(\cosh\delta_1\cosh\delta_2\cosh\delta_3\cosh\delta_4-\sinh\delta_1\sinh\delta_2\sinh\delta_3\sinh\delta_4).
\end{aligned}
$$

Note that we choose $G_N = \frac{\pi}{4}$.

2.2 Smarr's Formula

The entropy is given by $\frac{1}{4G_N}A$ where A is the area of the outer horizon. In this case, S has the form: [2]

$$\begin{aligned} S &= 16\pi[(m^2 + m\sqrt{m^2 - l^2})\,(\cosh\delta_1\cosh\delta_2\cosh\delta_3\cosh\delta_4) \\ &\quad +(m^2 - m\sqrt{m^2 - l^2})\,(\sinh\delta_1\sinh\delta_2\sinh\delta_3\sinh\delta_4)]. \end{aligned} \tag{5}$$

Note that we can write S in the form

$$S = 16\pi(m^2 K + m\sqrt{m^2 - l^2}L), \tag{6}$$

where

$$\begin{aligned} K &= \cosh\delta_1\,\cosh\delta_2\,\cosh\delta_3\,\cosh\delta_4 + \sinh\delta_1\,\sinh\delta_2\,\sinh\delta_3\,\sinh\delta_4, \\ L &= \cosh\delta_1\,\cosh\delta_2\,\cosh\delta_3\,\cosh\delta_4 - \sinh\delta_1\,\sinh\delta_2\,\sinh\delta_3\,\sinh\delta_4. \end{aligned} \tag{7}$$

We can write the variation of entropy in terms of the solution parameters as follows:

$$dS = \frac{\partial S}{\partial\delta_1}\,d\delta_1 + \frac{\partial S}{\partial\delta_2}\,d\delta_2 + \frac{\partial S}{\partial\delta_3}\,d\delta_3 + \frac{\partial S}{\partial\delta_4}\,\delta_4 + \frac{\partial S}{\partial m}\,dm + \frac{\partial S}{\partial l}\,dl, \tag{8}$$

But we want to write the variation in terms of the physical parameters:

$$dS = \Gamma_1\,dQ_1 + \Gamma_2\,dQ_2 + \Gamma_3\,dP_1 + \Gamma_4\,dP_4 + \Gamma_5\,dM + \Gamma_6\,dJ, \tag{9}$$

where

$$\Gamma_1 = \left(\frac{\partial S}{\partial\delta_1}\frac{\partial\delta_1}{\partial Q_1} + \frac{\partial S}{\partial\delta_2}\frac{\partial\delta_2}{\partial Q_1} + \frac{\partial S}{\partial\delta_3}\frac{\partial\delta_3}{\partial Q_1} + \frac{\partial S}{\partial\delta_4}\frac{\partial\delta_4}{\partial Q_1} + \frac{\partial S}{\partial m}\frac{\partial m}{\partial Q_1} + \frac{\partial S}{\partial l}\frac{\partial l}{\partial Q_1}\right), \tag{10}$$

etc.

By rearranging (9), we can now write the variation of M to obtain an analog of Smarr's formula:

$$dM = TdS + \Omega dJ + \Phi_1\,dQ_1 + \Phi_2\,dQ_2 + \Phi_3\,dP_1 + \Phi_4\,dP_2, \tag{11}$$

where

$$
\begin{aligned}
T &= \frac{1}{\Gamma_5}, \\
\Omega &= -\frac{\Gamma_6}{\Gamma_5}, \\
\Phi_i &= -\frac{\Gamma_i}{\Gamma_5} \quad (i = 1 \ldots 4).
\end{aligned} \tag{12}
$$

To determine the coefficients Γ_i, we have to invert the following matrix:

$$
\begin{pmatrix} dQ_1 \\ dQ_2 \\ dP_1 \\ dP_2 \\ dM \\ dJ \end{pmatrix} = \begin{pmatrix} 4mw_1 & 0 & 0 & 0 & 2z_1w_1 & 0 \\ 0 & 4mw_2 & 0 & 0 & 2z_2w_2 & 0 \\ 0 & 0 & 4mw_3 & 0 & 2z_3w_3 & 0 \\ 0 & 0 & 0 & 4mw_4 & 2z_4w_4 & 0 \\ 4mz_1w_1 & 4mz_2w_2 & 4mz_3w_3 & 4mz_4w_4 & \hat{M} & 0 \\ 8lmL_1 & 8lmL_2 & 8lmL_3 & 8lmL_4 & 8lL & 8mL \end{pmatrix} \begin{pmatrix} d\delta_1 \\ d\delta_2 \\ d\delta_3 \\ d\delta_4 \\ dm \\ dl \end{pmatrix} \tag{13}
$$

where

$$
\begin{aligned}
w_i &= \cosh 2\delta_i, \\
z_i &= \tanh 2\delta_i, \\
\hat{M} &= \frac{M}{m}, \\
L_i &= \frac{\partial L}{\partial \delta_i} \quad (i = 1, \ldots, 4).
\end{aligned} \tag{14}
$$

The result is

$$
\begin{pmatrix} d\delta_1 \\ d\delta_2 \\ d\delta_3 \\ d\delta_4 \\ dm \\ dl \end{pmatrix} = \frac{1}{B} \begin{pmatrix} \frac{B}{4mw_1} + z_1^2 & z_1z_2 & z_1z_3 & z_1z_4 & -z_1 & 0 \\ z_2z_1 & \frac{B}{4mw_2} + z_2^2 & z_2z_3 & z_2z_4 & -z_2 & 0 \\ z_3z_1 & z_3z_2 & \frac{B}{4mw_3} + z_3^2 & z_3z_4 & -z_3 & 0 \\ z_4z_1 & z_4z_2 & z_4z_3 & \frac{B}{4mw_4} + z_4^2 & -z_4 & 0 \\ -2mz_1 & -2mz_2 & -2mz_3 & -2mz_4 & 2m & 0 \\ \frac{u_1}{L} & \frac{u_2}{L} & \frac{u_3}{L} & \frac{u_4}{L} & \frac{lP_L}{L} - 2l & \frac{B}{8mL} \end{pmatrix} \begin{pmatrix} dQ_1 \\ dQ_2 \\ dP_1 \\ dP_2 \\ dM \\ dJ \end{pmatrix} \tag{15}
$$

where

$$
\begin{aligned}
u_i &= -z_i l P_L + 2 z_i l L - \frac{l L_i B}{4 m w_i} \ \ (i = 1, \dots, 4), \\
P_L &= L_1 z_1 + L_2 z_2 + L_3 z_3 + L_4 z_4, \\
B &= 2M - 4m(w_1 z_1^2 + w_2 z_2^2 + w_3 z_3^2 + w_4 z_4^2).
\end{aligned} \tag{16}
$$

Now, using this matrix, we can calculate the Γ_i's defined in (10).

$$
\begin{aligned}
\Gamma_5 &= \frac{S}{4\sqrt{m^2 - l^2}}, \\
\Gamma_6 &= \frac{-2\pi l}{\sqrt{m^2 - l^2}}, \\
\Gamma_i &= \frac{-S_i}{4\sqrt{m^2 - l^2}} \ \ (i = 1, ..., 4),
\end{aligned} \tag{17}
$$

where $S_i \equiv \frac{\partial S}{\partial \delta_i}$. Thus, we can find the coefficients in Smarr's formula (11) as: [1]

$$
\begin{aligned}
T &= \frac{4\sqrt{m^2 - l^2}}{S}, \\
\Omega &= \frac{8\pi l}{S}, \\
\Phi_i &= \frac{S_i}{S}.
\end{aligned} \tag{18}
$$

2.3 Thermodynamical Quantities Derived From the Metric

Now, we determine the thermodynamical quantities entering (11) using the metric. The temperature T is related to the surface gravity κ as

$$
2\pi T = \kappa = -\frac{1}{2} \frac{d g_{tt}}{dr} |_{r = r_+, \theta = 0}. \tag{19}
$$

[1] Related to the formula obtained in [6]

Using the metric

$$\frac{dg_{tt}}{dr}|_{r=r_+,\theta=0} = \frac{-2\,(r_+ - m)}{\sqrt{\Delta}}, \tag{20}$$

where

$$\sqrt{\Delta}|_{r=r_+,\theta=0} = 2m\,(m\,K + \sqrt{m^2 - l^2}\,L) = \frac{S}{8\pi}. \tag{21}$$

Thus,

$$\kappa = \frac{8\pi\sqrt{m^2 - l^2}}{S}, \tag{22}$$

which is in agreement with Smarr's formula (11).
The angular velocity of the black hole at the outer horizon is:

$$\Omega \equiv \frac{-g_{tt}}{g_{\phi t}}|_{r=r_+,\theta=0}. \tag{23}$$

From the metric (1) we can write

$$\frac{g_{tt}}{g_{\phi t}} = \frac{r^2 - 2mr + l^2 - l^2\sin^2\theta}{2ml\sin^2\theta\,(r\,L + m\,K - m\,L)}. \tag{24}$$

At the horizon, $r = r_+ = m + \sqrt{m^2 - l^2}$, which means $r^2 + l^2 - 2mr = 0$, so

$$\frac{g_{tt}}{g_{\phi t}}|_{r=r_+,\theta=0} = \frac{-l}{2m(\sqrt{m^2 - l^2}\,L + m\,K)} = \frac{-8\pi l}{S}, \tag{25}$$

$$\Omega_H = \frac{8\pi l}{S}, \tag{26}$$

which is also in agreement with Smarrs formula (11).

3 Black Holes in Five Dimensions

3.1 Metric and Physical Parameters

The metric for five-dimensional rotating charged black holes of toroidally compactified string theory, specified by the ADM mass M, three charges Q_1, Q_2, Q_3 and two rotational parameters l_1, l_2 is given by [3]:

$$
\begin{aligned}
ds_E^2 &= g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}d\theta^2 + 2g_{\phi\psi}d\phi d\psi + 2g_{\phi t}d\phi dt \\
&\quad +2g_{\psi t}d\psi dt + g_{\phi\phi}d\phi^2 + g_{\psi\psi}d\psi^2,
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
g_{tt} &= -\Delta^{\frac{-2}{3}}R(R-2m), \\
g_{rr} &= \frac{\Delta^{\frac{1}{3}}\,r^2}{(r^2+l_1^2)(r^2+l_2^2)-2mr^2}, \\
g_{\theta\theta} &= \Delta^{\frac{1}{3}}, \\
g_{\phi\psi} &= \cos^2\theta\sin^2\theta\Delta^{\frac{-2}{3}}(L_1k_3+L_3k_1), \\
g_{\phi t} &= -2m\sin^2\theta\Delta^{\frac{-2}{3}}(l_1Rc_1c_2c_3+l_2(2m-R)s_1s_2s_3), \\
g_{\psi t} &= -2m\cos^2\theta\Delta^{\frac{-2}{3}}(l_1(2m-R)s_1s_2s_3+l_2Rc_1c_2c_3), \\
g_{\phi\phi} &= \sin^2\theta\Delta^{\frac{-2}{3}}[\Delta+\sin^2\theta(L_1k_1+L_2k_2+L_3k_3)], \\
g_{\psi\psi} &= \cos^2\theta\Delta^{\frac{-2}{3}}[\Delta+\cos^2\theta(L_1k_1-L_2k_2+L_3k_3)],
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
L_1 &= l_1^2+l_2^2, \\
L_2 &= l_1^2-l_2^2, \\
L_3 &= 2l_1l_2, \\
k_1 &= mR-2m^2q-4m^2t, \\
k_2 &= R^2+mR+2mRp+2m^2q, \\
k_3 &= 4m^2\,c_1c_2c_3s_1s_2s_3, \\
R &= r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta, \\
p &= s_1^2+s_2^2+s_3^2, \\
q &= s_1^2s_2^2+s_1^2s_3^2+s_2^2s_3^2, \\
t &= s_1^2s_2^2s_3^2,
\end{aligned}
\tag{29}
$$

and $s_i,\ c_i$ stand for $\sinh\delta_i,\ \cosh\delta_i,\ (i=1,2,3)$ respectively.
Electromagnetic vector potentials are given as:

$$
\begin{aligned}
A^{(1)}_{t\,1} &= \frac{m\mathrm{cosh}\delta_1\mathrm{sinh}\delta_1}{r^2+2m\mathrm{sinh}^2\delta_1+l_1^2\mathrm{cos}^2\theta+l_2^2\mathrm{sin}^2\theta}, \\
A^{(1)}_{\phi\,1} &= m\mathrm{sin}^2\theta\frac{l_1\mathrm{sinh}\delta_1\mathrm{sinh}\delta_2\mathrm{cosh}\delta_3-l_2\mathrm{cosh}\delta_1\mathrm{cosh}\delta_2\mathrm{sinh}\delta_3}{r^2+2m\mathrm{sinh}^2\delta_1+l_1^2\mathrm{cos}^2\theta+l_2^2\mathrm{sin}^2\theta}, \\
A^{(1)}_{\psi\,1} &= m\mathrm{cos}^2\theta\frac{l_1\mathrm{cosh}\delta_1\mathrm{sinh}\delta_2\mathrm{sinh}\delta_3-l_2\mathrm{sinh}\delta_1\mathrm{cosh}\delta_2\mathrm{cosh}\delta_3}{r^2+2m\mathrm{sinh}^2\delta_1+l_1^2\mathrm{cos}^2\theta+l_2^2\mathrm{sin}^2\theta}, \\
A^{(2)}_{t\,1} &= \frac{m\mathrm{cosh}\delta_2\mathrm{sinh}\delta_2}{r^2+2m\mathrm{sinh}^2\delta_2+l_1^2\mathrm{cos}^2\theta+l_2^2\mathrm{sin}^2\theta}, \\
A^{(2)}_{\phi\,1} &= m\mathrm{sin}^2\theta\frac{l_1\mathrm{cosh}\delta_1\mathrm{sinh}\delta_2\mathrm{cosh}\delta_3-l_2\mathrm{sinh}\delta_1\mathrm{cosh}\delta_2\mathrm{sinh}\delta_3}{r^2+2m\mathrm{sinh}^2\delta_2+l_1^2\mathrm{cos}^2\theta+l_2^2\mathrm{sin}^2\theta}, \\
A^{(2)}_{\psi\,1} &= m\mathrm{cos}^2\theta\frac{l_1\mathrm{sinh}\delta_1\mathrm{cosh}\delta_2\mathrm{sinh}\delta_3-l_2\mathrm{cosh}\delta_1\mathrm{sinh}\delta_2\mathrm{cosh}\delta_3}{r^2+2m\mathrm{sinh}^2\delta_2+l_1^2\mathrm{cos}^2\theta+l_2^2\mathrm{sin}^2\theta}, \\
B_{t\phi} &= -2m\sin^2\theta(l_1\sinh\delta_1\sinh\delta_2\cosh\delta_3-l_2\cosh\delta_1\cosh\delta_2\sinh\delta_3)(r^2+l_1^2\cos^2\theta \\
&\quad +l_2^2\sin^2\theta+m\sinh^2\delta_1+m\sinh^2\delta_2)/[(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta+2m\sinh^2\delta_1) \\
&\quad \times(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta+2m\sinh^2\delta_2)], \\
B_{t\psi} &= -2m\cos^2\theta(l_2\sinh\delta_1\sinh\delta_2\cosh\delta_3-l_1\cosh\delta_1\cosh\delta_2\sinh\delta_3)(r^2+l_1^2\cos^2\theta \\
&\quad +l_2^2\sin^2\theta+m\sinh^2\delta_1+m\sinh^2\delta_2)/[(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta+2m\sinh^2\delta_1) \\
&\quad \times(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta+2m\sinh^2\delta_2)], \\
B_{\phi\psi} &= \frac{2m\cosh\delta_3\sinh\delta_3\cos^2\theta\sin^2\theta(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta+m\sinh^2\delta_1+m\sinh^2\delta_2)}{(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta+2m\sinh^2\delta_1)(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta+2m\sinh^2\delta_2)},
\end{aligned}
\tag{30}
$$

$$\Delta = R^3+2mpR^2+4m^2qR+8m^3t \tag{31}$$

$$r_\pm^2 = m-\frac{1}{2}L_1\pm\frac{1}{2}\sqrt{L_2^2+4m(m-L_1)} \tag{32}$$

We choose $G_N=\frac{\pi}{4}$.
The physical quantities ADM mass M, three charges Q_1, Q_2, Q_3 and two angular momenta J_1, J_2 are given as

$$
\begin{aligned}
M &= 2m(\cosh^2\delta_1+\cosh^2\delta_2+\cosh^2\delta_3)-3m, \\
Q_1 &= 2m\cosh\delta_1\,\sinh\delta_1, \\
Q_2 &= 2m\cosh\delta_2\,\sinh\delta_2,
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
Q_3 &= 2m \cosh\delta_3 \,\sinh\delta_3, \\
J_1 &= 4\,m\,(l_1\, \cosh\delta_1 \cosh\delta_2 \cosh\delta_3 - l_2\, \sinh\delta_1 \sinh\delta_2 \sinh\delta_3), \\
J_2 &= 4\,m\,(l_2\, \cosh\delta_1 \cosh\delta_2 \cosh\delta_3 - l_1\, \sinh\delta_1 \sinh\delta_2 \sinh\delta_3),
\end{aligned}
$$

where m is the nonextremality parameter, $\delta_{1,2,3}$ are the boost parameters and $l_{1,2}$ are the angular momentum parameters.

3.2 Smarr's Formula

The entropy is given by:[2]

$$
\begin{aligned}
S &= 4\pi m[\sqrt{2m-(l_1-l_2)^2}\,(\cosh\delta_1 \cosh\delta_2 \cosh\delta_3 + \sinh\delta_1 \sinh\delta_2 \sinh\delta_3) \\
&\quad + \sqrt{2m-(l_1+l_2)^2}(\cosh\delta_1 \cosh\delta_2 \cosh\delta_3 - \sinh\delta_1 \sinh\delta_2 \sinh\delta_3)]
\end{aligned}
\tag{34}
$$

$$
dS = \Gamma_1\, dQ_1 + \Gamma_2\, dQ_2 + \Gamma_3\, dQ_3 + \Gamma_4\, dM + \Gamma_5\, dJ_1 + \Gamma_6\, dJ_2. \tag{35}
$$

To find the derivatives of boosts with respect to physical variables ($\frac{\partial\delta_1}{\partial Q_2}$ etc.) we need to invert the following matrix:

$$
\begin{pmatrix} dQ_1 \\ dQ_2 \\ dQ_3 \\ dM \\ dJ_1 \\ dJ_2 \end{pmatrix}
=
\begin{pmatrix}
2mw_1 & 0 & 0 & z_1 w_1 & 0 & 0 \\
0 & 2mw_2 & 0 & z_2 w_2 & 0 & 0 \\
0 & 0 & 2mw_3 & z_3 w_3 & 0 & 0 \\
2mz_1w_1 & 2mz_2w_2 & 2mz_3w_3 & w_0+w_1+w_2 & 0 & 0 \\
J_{1,1} & J_{1,2} & J_{1,3} & J_1/m & 4mC & -4mE \\
J_{2,1} & J_{2,2} & J_{2,3} & J_2/m & -4mE & 4mC
\end{pmatrix}
\begin{pmatrix} d\delta_1 \\ d\delta_2 \\ d\delta_3 \\ dm \\ dl_1 \\ dl_2 \end{pmatrix}
\tag{36}
$$

where

$$
\begin{aligned}
w_i &= \cosh 2\delta_i \ \ (i=1,2,3), \\
z_i &= \tanh 2\delta_i \ \ (i=1,2,3), \\
C &= \cosh\delta_1\, \cosh\delta_2\, \cosh\delta_3, \\
E &= \sinh\delta_1\, \sinh\delta_2\, \sinh\delta_3.
\end{aligned}
\tag{37}
$$

The result is

$$
\begin{pmatrix} d\delta_1 \\ d\delta_2 \\ d\delta_3 \\ dm \\ dl_1 \\ dl_2 \end{pmatrix} = \frac{1}{2m\,U} \begin{pmatrix} \frac{U}{w_1}+z_1^2 & z_1 z_2 & z_1 z_3 & -z_1 & 0 & 0 \\ z_2 z_1 & \frac{U}{w_2}+z_2^2 & z_2 z_3 & -z_2 & 0 & 0 \\ z_3 z_1 & z_3 z_2 & \frac{U}{w_3}+z_3^2 & -z_3 & 0 & 0 \\ -2mz_1 & -2mz_2 & -2mz_3 & 2m & 0 & 0 \\ t_{51} & t_{52} & t_{53} & -c & CU/(2h) & EU/(2h) \\ t_{61} & t_{62} & t_{63} & -d & EU/(2h) & CU/(2h) \end{pmatrix} \begin{pmatrix} dQ_1 \\ dQ_2 \\ dQ_3 \\ dM \\ dJ_1 \\ dJ_2 \end{pmatrix} \tag{38}
$$

where

$$
\begin{aligned}
U &= (w_1 + w_2 + w_3 - w_1 z_1^2 - w_2 z_2^2 - w_3 z_3^2), \\
P_E &= E_1 z_1 + E_2 z_2 + E_3 z_3, \\
P_C &= C_1 z_1 + C_2 z_2 + C_3 z_3, \\
h &= (C - E)(C + E), \\
a &= \frac{E\,l_1 + C\,l_2}{h}, \\
b &= \frac{C\,l_1 + E\,l_2}{h}, \\
c &= 2l_1 + a\,P_E - b\,P_C, \\
d &= 2l_2 + b\,P_E - a\,P_C, \\
t_{5i} &= \frac{U}{w_i}(E_i\,a - C_i\,b) + z_i c, \\
t_{6i} &= \frac{U}{w_i}(E_i\,b - C_i\,a) + z_i d,
\end{aligned} \tag{39}
$$

Using these results, we can calculate $\Gamma_i' s$ as follows:

$$
\begin{aligned}
\Gamma_i &= \frac{-S_i}{\alpha\beta}, \\
\Gamma_4 &= \frac{S}{\alpha\beta}, && (40) \\
\Gamma_5 &= \pi\left(\frac{l_2 - l_1}{\alpha} - \frac{l_1 + l_2}{\beta}\right), && (41)
\end{aligned}
$$

$$\Gamma_6 = \pi\left(\frac{l_1 - l_2}{\alpha} - \frac{l_1 + l_2}{\beta}\right),$$

where

$$\begin{aligned}\alpha &= \sqrt{2m - (l_1 - l_2)^2}, \\ \beta &= \sqrt{2m - (l_1 + l_2)^2}.\end{aligned} \tag{42}$$

The Smarr's formula is of the form:

$$dM = T\,dS + \Phi_1\,dQ_1 + \Phi_2\,dQ_2 + \Phi_3\,dQ_3 + \Omega_1\,dJ_1 + \Omega_2\,dJ_2, \tag{43}$$

where

$$\begin{aligned}T &= \frac{\alpha\beta}{S}, \\ \Phi_i &= \frac{S_{,i}}{S}(i = 1, 2, 3), \qquad (44) \\ \Omega_1 &= -\pi\frac{\beta(l_2 - l_1) - \alpha(l_1 + l_2)}{S}, \\ \Omega_2 &= -\pi\frac{\beta(l_1 - l_2) - \alpha(l_1 + l_2)}{S}.\end{aligned}$$

$$(45)$$

3.3 Thermodynamic Quantities Derived from the Metric

Now, we make an independent check for the coefficients in Smarr's formula. Using the metric, we can calculate Ω_1.

$$\begin{aligned}\Omega_1 &\equiv \frac{-g_{tt}}{g_{\phi t}}\Big|_{r=r_+,\theta=\frac{\pi}{2}}, \qquad (46) \\ &= \frac{R(R - 2m)}{-2m\sin^2\theta(l_1 R c_1 c_2 c_3 + l_2(2m - R)s_1 s_2 s_3)}.\end{aligned}$$

where $R = r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta$. At the outer horizon, $r = r_+,\ \ \theta = \frac{\pi}{2}$.

$$R = -\frac{[\beta(l_2 - l_1) - \alpha(l_2 + l_1)]\, l_2}{\alpha - \beta}, \quad R - 2m = \frac{[\beta(l_2 - l_1) - \alpha(l_2 + l_1)] l_1}{\alpha + \beta} \tag{47}$$

So

$$\Omega_1 = -2\pi \frac{\beta(l_2 - l_1) - \alpha(l_2 + l_1)}{S} \tag{48}$$

This result is in agreement with Smarr's formula (43) except for a numerical factor. We can repeat the calculation for Ω_2. It is also in agreement with Smarr's formula. Now, let us make an independent check for κ, the surface gravity of the outer horizon. [8]

$$2\pi T = \kappa = -\frac{1}{2}\sqrt{-g^{rr}\, g^{tt}}\, \frac{d\Sigma}{dr}|_{r=r_+, \theta=\frac{\pi}{2}}, \tag{49}$$

where

$$\Sigma = g_{tt} - \frac{(g_{\phi t} + g_{\psi t})^2}{g_{\phi\phi} + g_{\psi\psi} + 2g_{\phi\psi}}. \tag{50}$$

We know that $g^{rr} = g_{rr}^{-1}$ and $g^{tt} = (g_{\phi\phi}g_{\psi\psi} - g_{\phi\psi}^2)/D$, where

$$\begin{aligned} D &= \frac{Det(g_{ij})}{g_{rr}g_{\theta\theta}} \\ D &= g_{tt}(g_{\phi\phi}g_{\psi\psi} - g_{\phi t}^2) + 2g_{\phi\psi}g_{\phi t}g_{\psi t} - g_{\psi t}^2 g_{\phi\phi} - g_{\phi t}^2 g_{\psi\psi} \end{aligned} \tag{51}$$

After some algebra, we find that

$$\begin{aligned} D &= \cos^2\theta \sin^2\theta[(2m - R)R + L_2(\cos^2\theta - \sin^2\theta)(R - m) - L_1 m + L_2^2 \cos^2\theta \sin^2\theta] \\ &= -\cos^2\theta \sin^2\theta\, \Delta^{\frac{1}{3}}\, r^2\, g_{rr}^{-1} \end{aligned} \tag{52}$$

$$\begin{aligned} g_{\phi\phi}g_{\psi\psi} - g_{\phi\psi}^2 &= \cos^2\theta \sin^2\theta \Delta^{-1/3}[\Delta + k_1 L_1 + (\sin^2\theta - \cos^2\theta)k_2 L_2 \\ &\quad + k_3 L_3 - \cos^2\theta \sin^2\theta L_2^2(2m + 2mp + R)] \end{aligned} \tag{53}$$

We know that $g^{rr} = g_{rr}^{-1}$, so

$$-g^{tt}\, g^{rr} = \frac{\Delta + k_1 L_1 + (\sin^2\theta - \cos^2\theta) k_2 L_2 + k_3 L_3 - \cos^2\theta \sin^2\theta L_2^2 (2m + 2mp + R)}{r^2\, \Delta^{\frac{2}{3}}} \tag{54}$$

$$\Sigma = \Delta^{\frac{1}{3}} \frac{R(2m - R) + R(\cos^2\theta - \sin^2\theta) L_2 - 2m(\cos^2\theta l_1 + \sin^2\theta l_2)^2}{\Delta + (L_1 k_1 + L_3 k_3)(\sin^4\theta + \cos^4\theta) + L_2 k_2 (\sin^2\theta - \cos^2\theta) + 2\sin^2\theta\cos^2\theta (L_1 k_3 + L_3 k_1)} \tag{55}$$

Note that at the horizon, $R = m - \frac{1}{2} L_2 + \frac{1}{2}\sqrt{L_2^2 + 4m(m - L_1)}$ and $\Sigma = 0$. We find

$$\kappa = \frac{\sqrt{l_2^2 + 4m(m - L_1)}}{\sqrt{\Delta + L_1 k_1 + L_2 k_2 + L_3 k_3}} \tag{56}$$

At the horizon, $S = 4\pi\sqrt{\Delta + L_1 k_1 + L_2 k_2 + L_3 k_3}$.

So,

$$\kappa = \frac{4\pi\alpha\beta}{S} \tag{57}$$

We can check the potentials Φ_i for the special case $l_1 = l_2 = 0$. In this case, $r_+^2 = 2m$, and $\Phi_1 = A_{t1}^{(1)}$, $\Phi_2 = A_{t1}^{(2)}$.

4 Conclusion

In this paper, we have calculated Smarr's formula for two different black holes, and then made an independent check using the metric coefficients. Temperature and angular momentum found in two different ways are in agreement.

Note that in four dimensions, we used the formula $\kappa = -\frac{1}{2}\frac{dg_{tt}}{dr}$, but in five dimensions we have to use the more general formula

$$\kappa = -\frac{1}{2}\sqrt{-g^{rr}\, g^{tt}}\, \frac{d}{dr}\left(g_{tt} - \frac{(g_{\phi t} + g_{\psi t})^2}{g_{\phi\phi} + g_{\psi\psi} + 2g_{\phi\psi}} \right)|_{r=r_+,\theta=\frac{\pi}{2}}, \tag{58}$$

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